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EXAMPLES OF DIFFERENT MINIMAL DIFFEOMORPHISMS GIVING THE SAME C*-ALGEBRAS

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ABSTRACT

We give examples of minimal diffeomorphisms of compact connected manifolds which are not topologically orbit equivalent, but whose transformation group C*-algebras are isomorphic. The examples show that the following properties of a minimal diffeomorphism are not invariants of the transformation group C*-algebra: having topologically quasidiscrete spectrum; the action on singular cohomology (when the manifolds are diffeomorphic); the homotopy type of the manifold (when the manifolds have the same dimension); and the dimension of the manifold.

These examples also give examples of nonconjugate isomorphic Cartan subalgebras, and of nonisomorphic Cartan subalgebras, of simple separable nuclear unital C*-algebras with tracial rank zero and satisfying the Universal Coefficient Theorem.

0. Introduction

The purpose of this paper is to give examples of distinct minimal diffeomorphisms of compact connected manifolds whose transformation group C*-algebras are isomorphic. This has become possible because of recent results which show that, in the real rank zero case, the transformation group C*-algebras of minimal diffeomorphisms are classifiable in the sense of the Elliott program [6].

For minimal homeomorphisms of the Cantor set, a remarkable theorem of Giordano, Putnam and Skau (Theorem 2.1 of [10]) asserts that isomorphism of

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the transformation group C*-algebras is equivalent to strong orbit equivalence of the homeomorphisms. On the circle, it is a consequence of the classification of the irrational rotation algebras that two minimal homeomorphisms have isomorphic crossed products if and only if they are flip conjugate. (See the beginning of Section 1 of [26] for details.)

Until now, nothing definite has been known about the problem on compact metric spaces other than the Cantor set and the circle. (There have been suggestive results, see Section 1 of [26] for an extensive discussion.) In this paper, we show that strong orbit equivalence is much too strong a relation to correspond to isomorphism of the C*-algebras. We give four kinds of examples of pairs of minimal diffeomorphisms of compact connected smooth manifolds which are not even topologically orbit equivalent, but which have isomorphic C*-algebras. These examples show that, for minimal diffeomorphisms of the same manifold, the action of the homeomorphism on singular cohomology and the property of having topologically quasidiscrete spectrum are not invariants of the C*-algebra; for minimal diffeomorphisms of manifolds of the same dimension, the homotopy type of the manifold is not an invariant of the C*-algebra; and for arbitrary minimal diffeomorphisms, the dimension of the manifold is not an invariant of the C*-algebra.

For minimal homeomorphisms of a connected compact metric space, strong orbit equivalence (even topological orbit equivalence) turns out to imply flip conjugacy. (See Theorem 3.1 and Remark 3.4 of [3] or Proposition 5.5 of [19].) Thus, our examples are equivalently examples of minimal diffeomorphisms which have isomorphic transformation group C*-algebras but which are not flip conjugate.

Our examples have another important consequence. They provide examples of simple separable nuclear unital C*-algebras, which even have tracial rank zero in the sense of [15], [16], with nonconjugate, even nonisomorphic, Cartan subalgebras. Cartan subalgebras (in the von Neumann algebra sense) of the hyperfinite type II_1 factor are unique up to automorphisms of the factor ([4]), although this is not true in a general type II_1 factor ([5]). Two slightly different definitions of Cartan subalgebras in C*-algebras have been introduced, in [31] and [14]. Corollary 1.16 in Chapter III of [31] gives conditions under which two Cartan subalgebras of a C*-algebra must be conjugate by an automorphism. (The conditions are restrictive: among other things, both subalgebras must be AF. It isn't part of the hypotheses, but one sees from the proof that the C*-algebra must also be AF.)

The existence of nonconjugate Cartan subalgebras is not new: see the end of the introduction to [13] for nonisomorphic Cartan subalgebras, and see Theorem 1.13 below, which is immediate from results already in the literature, for isomorphic but nonconjugate Cartan subalgebras. Our examples give both simple nuclear C*-algebras with new kinds of isomorphic but nonconjugate Cartan subalgebras, and simple nuclear C*-algebras with many Cartan subalgebras with quite different structure, being algebras of continuous functions on compact spaces of different dimensions.

All our examples are smooth. Since the announcement of our results (Section 2 of [26]), an easier proof of the required classification results has been found [18], which also does not require smoothness. This has made possible the last three examples in Section 5 of [18], which give further examples of pairs of minimal homeomorphisms with isomorphic transformation group C*-algebras but which are not topologically orbit equivalent. Two of these examples, both discussed in Section 1 of [26] but with the isomorphisms only conjectured, involve zero and one dimensional compact metric spaces which are highly disconnected. The third, based on [11], involves connected one dimensional spaces which are not locally connected.

In this paper, we could omit some of the work by ignoring smoothness. We prefer to keep smoothness, because we believe that the problem of finding conditions on minimal diffeomorphisms for the isomorphism of smooth crossed products is important. In particular, this problem might have an answer quite different from the problem for C* crossed products. See Section 3 of [26] for more. We only note here that for the smooth crossed product algebra to exist, a kind of temperedness condition must be satisfied.

To summarize, besides providing examples of different minimal homeomorphisms with isomorphic transformation group C*-algebras, there are three ways one can think of the results:

- Properties of minimal homeomorphisms which are not invariants of the transformation group C*-algebra.
- Examples of nonconjugate, sometimes nonisomorphic Cartan subalgebras.
- Examples on which to test the question of whether smooth crossed products preserve more information than C* crossed products.

We may also think of the paper as an application of H. Lin's classification theorem [17].

This paper is organized as follows. In Section 1, we state in a convenient form some general results that will be repeatedly used. Each of the remaining four sections is devoted to one example.

I am grateful to Qing Lin for suggesting a nice generalization of the original version of the second example.

1. Preliminaries

In this section, we present, for convenience, some results which will be used repeatedly in the rest of the paper. They are the computation of the ordered K-theory of crossed products, using the Pimsner–Voiculescu exact sequence and Exel's rotation numbers, classification for crossed products by minimal diffeomorphisms, some basic facts about flow equivalence, and some basic facts about Cartan subalgebras of crossed products. Nothing here is really new.

For reference, we state here the Pimsner–Voiculescu exact sequence [29] for the special case of a crossed product of a compact space by a homeomorphism. This is what we use to compute unordered K-theory of crossed products.

THEOREM 1.1: *Let X be a compact Hausdorff space, and let $h\colon X \to X$ be a homeomorphism. Then there is a natural 6 term exact sequence*

$$\begin{array}{ccccc} K^0(X) & \xrightarrow{\ \mathrm{id} - h^*\ } & K^0(X) & \longrightarrow & K_0(C^*(\mathbf{Z}, X, h)) \\ {\scriptstyle \exp}\uparrow & & & & \downarrow{\scriptstyle \partial} \\ K_1(C^*(\mathbf{Z}, X, h)) & \longleftarrow & K^1(X) & \xleftarrow{\ \mathrm{id} - h^*\ } & K^1(X) \end{array} .$$

The maps $K^j(X) \to K_j(C^*(\mathbf{Z}, X, h))$ are from the inclusion $C(X) \to C^*(\mathbf{Z}, X, h)$.

To get the sequence in this form, take the action on $C(X)$ to be $\alpha(f) = f \circ h^{-1}$, and identify $K_*(C(X))$ with $K^*(X)$ and $\alpha_*\colon K_*(C(X)) \to K_*(C(X))$ with $(h^{-1})^*\colon K^*(X) \to K^*(X)$.

Since we will make frequent use of it, we recall the machinery of [7] for the computation of the map on K_0 of a crossed product by a homeomorphism determined by an invariant measure.

Definition 1.2: Let X be a connected compact metric space, and let $h\colon X \to X$ be a homeomorphism. Let $A = C^*(\mathbf{Z}, X, h)$ be the transformation group C*-algebra. Let μ be an h-invariant Borel probability measure on X. Let

$$K^1(X)^h = \{\eta \in K^1(X)\colon h^*(\eta) = \eta\}.$$

Define $\rho_h^\mu\colon K^1(X)^h \to S^1$ as follows. Given $u \in U(C(X, M_n))$ such that $[u \circ h^{-1}] = [u]$, set $z(x) = \det(u(x))$, find a continuous function $a\colon X \to \mathbf{R}$ such that $z(h^{-1}(x))^* z(x) = \exp(2\pi i a(x))$, and define

$$\rho_h^\mu([u]) = \exp\left(2\pi i \int_X a\, d\mu\right).$$

The number $\rho_h^\mu([u])$ is called the **rotation number** of $[u]$ with respect to h and μ.

The main result that we use is the following theorem. It is obtained by combining Definition IV.1 and Theorems V.12 and VI.11 of [7], using the definitions and properties of the maps Det_* and R_h^μ of Theorem VI.11 of [7], which are contained in Section VI of [7]. (By comparing Definition VI.2 with the proof of Proposition VI.10 in [7], one sees that the automorphism of $C(X)$ given by h really is $\alpha(f) = f \circ h^{-1}$.)

Theorem 1.3 ([7]): *Let X, h, A and μ be as in Definition 1.2. The function ρ_h^μ: $K^1(X)^h \to S^1$ of Definition 1.2 is a well-defined group homomorphism. Moreover, if ∂: $K_0(A) \to K^1(X)$ is the connecting homomorphism in the Pimsner–Voiculescu exact sequence, τ: $A \to \mathbf{C}$ is the tracial state on A determined by μ, and $\eta \in K_0(A)$, then $\partial(\eta) \in K^1(X)^h$ and $\exp(2\pi i \tau_*(\eta)) = \rho_h^\mu \circ \partial(\eta)$.*

Note that by combining the Pimsner–Voiculescu exact sequence, Exel's results, and Theorem 4.5(1) of [27], we have machinery for the complete computation of the Elliott invariant of the transformation group C*-algebra of a minimal homeomorphism of a finite dimensional connected compact metric space.

The following result is a special case of results of [21], and is what we use to establish isomorphisms of crossed products.

Theorem 1.4: *For $k = 1, 2$ let M_k be a compact smooth manifold, and let h_k: $M_k \to M_k$ be a uniquely ergodic minimal diffeomorphism. Let τ_k be the unique tracial state on $C^*(\mathbf{Z}, M_k, h_k)$. Assume that there are isomorphisms*

$$f_j\colon K_j(C^*(\mathbf{Z}, M_1, h_1)) \to K_j(C^*(\mathbf{Z}, M_2, h_2)),$$

such that $f_0([1]) = [1]$ and $(\tau_2)_* \circ f_0 = (\tau_1)_*$, as maps from $K_0(C^*(\mathbf{Z}, M_1, h_1))$ to $\mathbf{R}$. Suppose further that $(\tau_1)_*$ has dense range. Then $C^*(\mathbf{Z}, M_1, h_1) \cong C^*(\mathbf{Z}, M_2, h_2)$.

Proof: By the main result of [21] (also see the survey article [20]), the crossed products are direct limits, with no dimension growth, of recursive subhomogeneous algebras. Theorem 2.3 of [27] therefore shows that the order on K_0 is determined by the tracial states: $\eta \in K_0(C^*(\mathbf{Z}, M_j, h_j))$ is positive if and only if either $\eta = 0$ or $(\tau_j)_*(\eta) > 0$. So f_0 is an order isomorphism. The rest of the hypotheses now implies that the Elliott invariants of the two crossed products are isomorphic. Since there is a unique tracial state, the hypothesis that $(\tau_1)_*$ have dense range implies that, for $A = C^*(\mathbf{Z}, M_1, h_1)$, the canonical map $\rho_A\colon K_0(A) \to \mathrm{Aff}(T(A))$ has image $\rho_A(K_0(A))$ dense in $\mathrm{Aff}(T(A))$. Hence the same is true for $A = C^*(\mathbf{Z}, M_2, h_2)$. Therefore the crossed products have tracial rank zero by Theorem 4.4 of [28]. As in the proof of Theorem 4.5 of [28], the classification theorem, Theorem 5.2 of [17], applies, and $C^*(\mathbf{Z}, M_1, h_1) \cong C^*(\mathbf{Z}, M_2, h_2)$. ∎

We can also use Corollary 4.8 of [18] in place of [21] and [28].

We now turn to flow equivalence, a weaker relation than conjugacy. Since the relevant relation for our purposes is actually flip conjugacy, we will actually introduce and use flip flow equivalence. Flip flow equivalence does not imply isomorphism of the crossed products, only Morita equivalence (Lemma 1.2 of [23]; Situation 8 of [32]). Indeed, flip flow equivalence seems to play the same role for Morita equivalence that flip conjugacy does for isomorphism. (Theorem 2.6 of [10], based on Definition 1.8 of [10], is at least suggestive, although in a different context.) Thus, we give flip flow equivalence less emphasis. However, it seems inappropriate to ignore it entirely.

The following definitions (except flip flow equivalence) can be found, for example, at the beginning of Section 1 of [24] and in Definition 1.1 and the following discussion in [23].

Definition 1.5: A **cross section** of an action of $\mathbf{R}$ on a compact metric space X is a closed subset $K \subset X$ such that the map from $\mathbf{R} \times K$ to X, given by $(t, x) \mapsto tx$, is a surjective local homeomorphism.

We always take the cross section K to be equipped with the homeomorphism $h\colon K \to K$ defined as follows. For $x \in K$, let $\lambda_K(x) \in \mathbf{R}$ be given by

$$\lambda_K(x) = \inf(\{t \in (0, \infty)\colon t \cdot x \in K\}),$$

and take $h(x) = \lambda_K(x) \cdot x$.

Theorem 1 of [35] gives some equivalent conditions for K to be a cross section.

Definition 1.6: Homeomorphisms $h_1\colon X_1 \to X_1$ and $h_2\colon X_2 \to X_2$ are **flow equivalent** if both can be obtained as cross sections to the same flow. They are **flip flow equivalent** if h_1 is flow equivalent to h_2 or to h_2^{-1}.

One can simplify a little (still following Section 1 of [24]):

Definition 1.7: Let X be a topological space, and let $h\colon X \to X$ be a homeomorphism. The **suspension** of (X, h) is the flow (action of $\mathbf{R}$ on a topological space) constructed as follows. On the space $X \times \mathbf{R}$, we define an action of $\mathbf{R}$

by $s \cdot (x, t) = (x, s + t)$, and an action of $\mathbf{Z}$ by taking the generator to be the homeomorphism $(x, t) \mapsto (h(x), t - 1)$. These actions commute, and therefore the action of $\mathbf{R}$ on the space $Y = (X \times \mathbf{R})/\mathbf{Z}$ is well-defined and continuous. We define the suspension of (X, h) to be Y equipped with this action of $\mathbf{R}$.

In [24], this flow is called **the mapping torus flow**.

LEMMA 1.8: *Homeomorphisms $h_1\colon X_1 \to X_1$ and $h_2\colon X_2 \to X_2$ are flow equivalent if and only if (X_2, h_2) can be obtained as a cross section of the suspension of (X_1, h_1).*

Finally, we state some results on Cartan subalgebras. The following version of a Cartan subalgebra is taken from Definitions 1 and 3 in Section 1 of [14].

Definition 1.9: Let A be a unital C*-algebra. A **normalizer** of a C*-subalgebra B of A is an element $a \in A$ such that $aBa^* \subset B$ and $a^*Ba \subset B$. It is **free** if $a^2 = 0$. A **diagonal** in A is a commutative C*-subalgebra B of A, which contains 1_A and such that there is a faithful conditional expectation $P\colon A \to B$ whose kernel is the closed linear span of the free normalizers of B.

By Proposition 4 in Section 1 of [14], the definition implies that a diagonal is a maximal abelian subalgebra. In the next theorem, we are interested in minimal homeomorphisms of infinite compact metric spaces, but the proof is the same in greater generality. One should be able to obtain the result as a corollary of results in [14], but a direct proof seems more illuminating.

THEOREM 1.10: *Let X be a compact Hausdorff space with a free action of a discrete amenable group Γ. Then $C(X)$ is a diagonal in $C^*(\Gamma, X)$ in the sense of Definition 1.9.*

Proof: The conditional expectation is the standard one: letting $u_\gamma \in C^*(\Gamma, X)$ be the standard unitary corresponding to $\gamma \in \Gamma$, if all but finitely $f_\gamma \in C(X)$ are zero, then $P(\sum_{\gamma\in\Gamma} f_\gamma u_\gamma) = f_1$. The only part requiring proof is that its kernel is the closed linear span of the free normalizers of $C(X)$.

First, let $a \in C^*(\Gamma, X)$ be a free normalizer; we prove that $P(a) = 0$. The definition of a normalizer implies, in particular, that $a^*a \in C(X)$. Using the properties of conditional expectations at the second step and $a^2 = 0$ at the third, we get

$$[aP(a)]^*[aP(a)] = P(a)^*a^*aP(a) = P(a^*)P((a^*a)a) = 0,$$

whence $aP(a) = 0$. Using $P(a) \in C(X)$, we now get $P(a)^2 = P(aP(a)) = 0$. Since $P(a)$ is an element of a commutative C*-algebra, this implies that $P(a) = 0$.

We finish the proof by showing that the closed linear span L of the free normalizers contains $\mathrm{Ker}(P)$. Since finite sums of the form $\sum_{\gamma\in\Gamma} f_\gamma u_\gamma$ are dense in $C^*(\Gamma, X)$, and since

$$\mathrm{Ker}(P) = \{a - P(a) \colon a \in C^*(\Gamma, X)\},$$

it suffices to show that $f u_\gamma \in L$ for every $f \in C(X)$ and $\gamma \in \Gamma \setminus \{1\}$. Since X is compact and the group element γ has no fixed points, there is a finite cover of X consisting of open sets $U \subset X$ such that $\gamma U \cap U = \emptyset$. Choose a partition of unity $(g_1, g_2, \ldots, g_n)$ subordinate to such a cover. Then $f u_\gamma = \sum_{k=1}^{n} f g_k u_\gamma$, we have

$$(f g_k u_\gamma)^2 = f(f \circ \gamma^{-1}) g_k (g_k \circ \gamma^{-1}) u_{\gamma^2} = 0$$

because $\mathrm{supp}(g_k) \cap \mathrm{supp}(g_k \circ \gamma^{-1}) = \emptyset$, and it is trivial that $f g_k u_\gamma$ normalizes $C(X)$. Thus $f u_\gamma \in L$. ∎

The same result also holds for Renault's definition of a Cartan subalgebra.

THEOREM 1.11: *Let X be an infinite compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. Then $C(X)$ is a Cartan subalgebra of $C^*(\mathbf{Z}, X, h)$ in the sense of Definition 4.13 in Chapter II of [31].*

Proof: This is immediate from Proposition 4.14 in Chapter II of [31]. ∎

For the application to our examples, we combine the above with a result of Tomiyama.

PROPOSITION 1.12: *Let $h_1 \colon X_1 \to X_1$ and $h_2 \colon X_2 \to X_2$ be minimal homeomorphisms of infinite compact metric spaces such that h_1 and h_2 are not flip conjugate. Suppose there is an isomorphism $\varphi \colon C^*(\mathbf{Z}, X_1, h_1) \to C^*(\mathbf{Z}, X_2, h_2)$. Then $\varphi(C(X_1))$ and $C(X_2)$ are diagonals in $C^*(\mathbf{Z}, X_2, h_2)$, and are Cartan subalgebras in the sense of Renault, which are not conjugate by any automorphism of $C^*(\mathbf{Z}, X_2, h_2)$.*

Proof: That $\varphi(C(X_1))$ and $C(X_2)$ are diagonals is Theorem 1.10, and that they are Cartan subalgebras is Theorem 1.11. If there is an automorphism ψ of $C^*(\mathbf{Z}, X_2, h_2)$ such that $\psi(\varphi(C(X_1))) = C(X_2)$, then

$$\psi \circ \varphi \colon C^*(\mathbf{Z}, X_1, h_1) \to C^*(\mathbf{Z}, X_2, h_2)$$

is an isomorphism sending $C(X_1)$ to $C(X_2)$, so the corollary at the end of [39] implies that h_1 and h_2 are flip conjugate. ∎

We can use results already in the literature to give examples of simple unital AT algebras with real rank zero which have uncountably many isomorphic but nonconjugate Cartan subalgebras.

Theorem 1.13: *Let A be a simple unital AT algebra with real rank zero and with $K_1(A) \cong \mathbf{Z}$. Then there exist uncountably many diagonals (Cartan subalgebras) $B_t \subset A$, for $t \in [0, \infty]$, such that $B_s \cong B_t$ for all s and t, but such that for $s \neq t$ there is no automorphism φ of A with $\varphi(B_s) = B_t$.*

Proof: It is not possible to have $K_0(A) \cong \mathbf{Z}$. So Theorem 1.15 of [10] provides a minimal homeomorphism h of the Cantor set X such that $A \cong C^*(\mathbf{Z}, X, h)$. Use Theorem 2.3 of [2], Theorem 7.1 of [38] and Theorem 6.1 of [37] to choose, for every $t \in [0, \infty]$, a minimal homeomorphism h_t of X which is strong orbit equivalent to h and which has topological entropy equal to t. Theorem 2.1 of [10] provides isomorphisms ψ_t: $C^*(\mathbf{Z}, X, h_t) \to A$. Set $B_t = \psi_t(C(X))$. For $s \neq t$, the homeomorphisms h_s and h_t are not flip conjugate because they have different topological entropies. So Proposition 1.12 implies that B_s and B_t are diagonals (Cartan subalgebras) which are not conjugate by an automorphism of A. ∎

Applying the isomorphism result of Example 5.8 of [18] in the same way, we see that each such algebra also has at least one diagonal (Cartan subalgebra) which is not isomorphic to the ones in the theorem.

2. First example: Furstenberg transformations on $(S^1)^2$

We give two Furstenberg transformations on the 2-torus $S^1 \times S^1$ with isomorphic transformation group C*-algebras, such that one has topologically quasidiscrete spectrum and the other does not. In terms of the three additional ways to think of the results:

- Having topologically quasidiscrete spectrum or not is not an invariant of the transformation group C*-algebra.
- The C*-algebra in the example has two isomorphic diagonals (Cartan subalgebras) which are the algebras of continuous functions on a connected space and which are not conjugate by an automorphism of the algebra.
- Both diffeomorphisms are tempered in the sense of Definition 3.1 of [26] (see Example 3.7 of [26]), and the actions are very similar, so isomorphism of the smooth crossed products is a very interesting problem.

Example 2.1: Let $\theta \in [0, 1] \setminus \mathbf{Q}$ be an irrational number, and let r: $S^1 \to \mathbf{R}$ be a smooth function, both chosen according to the proof of Lemma 2.3 of [33].

(See below for details.) Define $h_1, h_2\colon S^1 \times S^1 \to S^1 \times S^1$ by

$$h_1(\zeta_1, \zeta_2) = (e^{2\pi i\theta}\zeta_1, \zeta_1\zeta_2) \quad \text{and} \quad h_2(\zeta_1, \zeta_2) = (e^{2\pi i\theta}\zeta_1, e^{2\pi i r(\zeta_1)}\zeta_1\zeta_2)$$

for $(\zeta_1, \zeta_2) \in S^1 \times S^1$. (The only difference is the extra factor $\exp(2\pi i r(\zeta_1))$ in the definition of h_2.)

With these choices, it is proved in [33] that h_1 has topologically quasidiscrete spectrum (see Section 1 of [33] for the definition), while h_2 does not. Therefore h_1 is not flip conjugate to h_2. The Elliott invariants of the transformation group C*-algebras for all such homomorphisms are computed in Example 4.9 of [27], and in particular it is shown there that for uniquely ergodic homeomorphisms of this form, the Elliott invariants are all isomorphic, and that the unique tracial state τ has $\tau_*(K_0(C^*(\mathbf{Z}, S^1 \times S^1, h)))$ dense in $\mathbf{R}$. Now h_2 is uniquely ergodic (see the proof of Theorem 2.1 of [33]), and h_1 is uniquely ergodic and both h_1 and h_2 are minimal (see [33] or Example 4.9 of [27]; the original source is Section 2 of [9]). Therefore, to use Theorem 1.4 to prove that $C^*(\mathbf{Z}, S^1 \times S^1, h_1) \cong C^*(\mathbf{Z}, S^1 \times S^1, h_2)$, it suffices to verify that r is smooth.

Recall from the proof of Lemma 2.3 of [33] that $\nu_1 = 1$, that $\nu_{k+1} = 2^{\nu_k} + \nu_k + 1$ for $k \geq 1$, and that $n_k = \operatorname{sgn}(k) \cdot 2^{\nu_{|k|}}$ for $k \in \mathbf{Z} \setminus \{0\}$. Further recall that

$$\theta = \sum_{k=1}^{\infty} 2^{-\nu_k}$$

and

$$r(t) = \sum_{k \in \mathbf{Z} \setminus \{0\}} \beta_k e^{2\pi i n_k t},$$

with

$$\beta_k = \frac{1}{|k|}(e^{2\pi i n_k \theta} - 1).$$

It follows from the proof in [33] that $|\beta_k| \leq 2\pi \cdot 2^{-|n_k|} \cdot |k|^{-1}$ for all $k \neq 0$. To prove that r is smooth, it is enough to prove uniform convergence, for every $m \geq 0$, of the series

$$\sum_{k \in \mathbf{Z} \setminus \{0\}} (2\pi i n_k)^m \beta_k e^{2\pi i n_k t},$$

obtained by differentiating the series for $r(t)$ term by term m times. For $k \geq 1$ the n_k are distinct positive integers, so

$$\sum_{k \in \mathbf{Z} \setminus \{0\}} |(2\pi i n_k)^m \beta_k| \leq 2(2\pi)^{m+1} \sum_{k=1}^{\infty} \frac{n_k^m}{2^{n_k} k} \leq 2(2\pi)^{m+1} \sum_{n=1}^{\infty} \frac{n^m}{2^n} < \infty.$$

Since the functions $t \mapsto e^{2\pi i n_k t}$ have absolute value 1, this proves the required uniform convergence.

It now follows from Theorem 1.4 that $C^*(\mathbf{Z}, S^1 \times S^1, h_1) \cong C^*(\mathbf{Z}, S^1 \times S^1, h_2)$.

From Proposition 1.12, we see that the $C^*(\mathbf{Z}, S^1 \times S^1, h_2)$ has two nonconjugate diagonals (Cartan subalgebras), both isomorphic to $C(S^1 \times S^1)$.

The example answers the isomorphism question raised in [33] before Proposition 2.5, but in the opposite way to what is suggested there.

It is likely that the diffeomorphisms h_1 and h_2 can be shown not to be flip flow equivalent (Definition 1.6), by using Theorem 4.1 of [24].

One intriguing question arises.

QUESTION 2.2: *Let h_1 and h_2 be as in Example 2.1. Does there exist a minimal homeomorphism h of $S^1 \times S^1$ such that both h_1 and h_2 are factors of h?*

3. Second example: Affine Furstenberg transformations on $(S^1)^3$

We give two affine Furstenberg transformations on $(S^1)^3$ whose C*-algebras are isomorphic but which are not flip conjugate. Variations produce an arbitrarily large number of such transformations. The isomorphism of the C*-algebras answers a question raised in Section 6.1 of the unpublished thesis of R. Ji [12]. We point out here that there is a mistake in [12], which is identified and corrected in follow-up work [30]. In particular, we refer to [30] for the best current knowledge of the (unordered) K-theory of transformation group C*-algebras of Furstenberg transformations in high dimensions. The mistake in [12] does not affect the examples in this section.

In terms of our three additional ways to think of the results:

- The action of the homeomorphism on integral cohomology not an invariant of the transformation group C*-algebra.
- There are C*-algebras with arbitrarily large finite numbers of isomorphic diagonals (Cartan subalgebras) which are the algebras of continuous functions on a connected space and which are not conjugate by automorphisms of the algebra.
- All the diffeomorphisms are tempered in the sense of Definition 3.1 of [26] (see Example 3.6 of [26]), and the actions are very similar, so isomorphism of the smooth crossed products is a very interesting problem.

We begin with a general calculation for affine Furstenberg transformations on $(S^1)^3$. Most of it is a special case of computations done in [12]; the only new part is the determination of the order on the K-theory of the crossed product, rather than just showing what the (unique) tracial state does on K-theory. A similar calculation for $(S^1)^2$ was done in Example 4.9 of [27]. We give the calculation here because [12] has never been published, and because there are new features in this case which do not appear in the calculation of [27]. Also, we need this result for our third example.

LEMMA 3.1: *Let $M = (S^1)^3$, let $m, n \in \mathbf{Z} \setminus \{0\}$, and define $h = h_{m,n,\theta}\colon M \to M$ by*

$$h(\zeta_1, \zeta_2, \zeta_3) = (\exp(2\pi i\theta)\zeta_1, \zeta_1^m\zeta_2, \zeta_2^n\zeta_3), \quad \text{for } (\zeta_1, \zeta_2, \zeta_3) \in (S^1)^3.$$

Then h is minimal and uniquely ergodic. Moreover, $K_0(C^*(\mathbf{Z}, M, h))$ and $K_1(C^*(\mathbf{Z}, M, h))$ are both isomorphic to $\mathbf{Z}^4 \oplus \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$. The isomorphism of $K_0(C^*(\mathbf{Z}, M, h))$ with this group can be chosen in such a way that the unique tracial state τ induces the map

$$\tau_*(r_1, r_2, r_3, r_4, s_1, s_2) = r_1 + \theta r_3$$

and $K_0(C^*(\mathbf{Z}, M, h))_+$ is identified with

$$\{(r_1, r_2, r_3, r_4, s_1, s_2) \in \mathbf{Z}^4 \oplus \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}\colon r_1 + r_3\theta > 0\} \cup \{0\}.$$

Proof: That h is minimal and uniquely ergodic follows from Theorem 2.1 in Section 2.3 of [9]. By this same theorem, the unique ergodic measure is the normalized Lebesgue measure μ on $(S^1)^3$.

The Künneth Theorem (Theorem 4.1 of [34]; also see Corollary 2.7.15 of [1] for the commutative case, which suffices here) shows that

$$K^*((S^1)^3) \cong K^*(S^1) \otimes K^*(S^1) \otimes K^*(S^1).$$

We identify the two sides of this isomorphism. Let $1 \in C(S^1)$ be the identity, and let $z \in C(S^1)$ be the canonical unitary $z(\zeta) = \zeta$. Then $K^0((S^1)^3)$ is the free abelian group on generators

$$\eta_1 = [1] \otimes [1] \otimes [1], \quad \eta_2 = [z] \otimes [z] \otimes [1],$$

$$\eta_3 = [z] \otimes [1] \otimes [z] \quad \text{and} \quad \eta_4 = [1] \otimes [z] \otimes [z],$$

and $K^1((S^1)^3)$ is the free abelian group on generators

$$\gamma_1 = [z] \otimes [1] \otimes [1], \quad \gamma_2 = [1] \otimes [z] \otimes [1],$$

$$\gamma_3 = [1] \otimes [1] \otimes [z] \quad \text{and} \quad \gamma_4 = [z] \otimes [z] \otimes [z].$$

Moreover, in the graded ring structure on $K^*((S^1)^3)$, we have

$$\eta_2 = \gamma_1\gamma_2, \quad \eta_3 = \gamma_1\gamma_3, \quad \eta_4 = \gamma_2\gamma_3, \quad \gamma_4 = \gamma_1\gamma_2\gamma_3 \quad \text{and} \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 0.$$

To compute h^*, it therefore suffices to calculate $h^*(\gamma_1)$, $h^*(\gamma_2)$ and $h^*(\gamma_3)$. We may replace h by the homotopic map

$$h_0(\zeta_1, \zeta_2, \zeta_3) = (\zeta_1, \zeta_1^m \zeta_2, \zeta_2^n \zeta_3).$$

Then $h^*(\gamma_1)$ is the class of the function

$$(z \otimes 1 \otimes 1) \circ h_0 = z \otimes 1 \otimes 1,$$

so $h^*(\gamma_1) = \gamma_1$. Similarly,

$$h^*(\gamma_2) = [(1 \otimes z \otimes 1) \circ h_0] = [z^m \otimes z \otimes 1] = m\gamma_1 + \gamma_2 \quad \text{and} \quad h^*(\gamma_3) = n\gamma_2 + \gamma_3.$$

It follows that

$$h^*(\eta_2) = \gamma_1(m\gamma_1 + \gamma_2) = \eta_2 \quad (\text{using the fact that } \gamma_1^2 = 0),$$
$$h^*(\eta_3) = \gamma_1(n\gamma_2 + \gamma_3) = n\eta_2 + \eta_3$$

and

$$h^*(\eta_4) = (m\gamma_1 + \gamma_2)(n\gamma_2 + \gamma_3) = mn\eta_2 + m\eta_3 + \eta_4,$$

and that $h^*(\gamma_4) = \gamma_4$. So the matrices of $\mathrm{id} - h^*$ on $K^0((S^1)^3)$ and $K^1((S^1)^3)$ are given by

$$0 \oplus \begin{pmatrix} 0 & -n & -mn \\ 0 & 0 & -m \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -m & 0 \\ 0 & 0 & -n \\ 0 & 0 & 0 \end{pmatrix} \oplus 0.$$

Thus

$$\mathrm{id} - h^* \colon K^0((S^1)^3) \to K^0((S^1)^3)$$

has kernel $\mathbf{Z}\eta_1 + \mathbf{Z}\eta_2$ and cokernel $\mathbf{Z}\overline{\eta}_1 + \mathbf{Z}\overline{\eta}_2 + \mathbf{Z}\overline{\eta}_3 + \mathbf{Z}\overline{\eta}_4$, in which the images $\overline{\eta}_1$ and $\overline{\eta}_4$ of η_1 and η_4 have infinite order, in which $\overline{\eta}_2$ has order n, and in which $\overline{\eta}_3$ has order m. Similarly,

$$\mathrm{id} - h^* \colon K^1((S^1)^3) \to K^1((S^1)^3)$$

has kernel $\mathbf{Z}\gamma_1 + \mathbf{Z}\gamma_4$ and cokernel isomorphic to $\mathbf{Z}^2 \oplus \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$.

The exact sequence of Theorem 1.1 therefore breaks apart into the two exact sequences

$$0 \longrightarrow [\mathbf{Z}\overline{\eta}_1 + \mathbf{Z}\overline{\eta}_4] \oplus \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z} \longrightarrow K_0(C^*(\mathbf{Z}, M, h)) \xrightarrow{\partial} \mathbf{Z}\gamma_1 + \mathbf{Z}\gamma_4 \longrightarrow 0$$

and

$$0 \longrightarrow \mathbf{Z}^2 \oplus \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z} \longrightarrow K_1(C^*(\mathbf{Z}, M, h)) \longrightarrow \mathbf{Z}^2 \longrightarrow 0.$$

Both split because $\mathbf{Z}^2$ is free. Therefore $K_0(C^*(\mathbf{Z}, M, h))$ and $K_1(C^*(\mathbf{Z}, M, h))$

are both isomorphic to $\mathbf{Z}^4 \oplus \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$, as claimed. We are now done with $K_1(C^*(\mathbf{Z}, M, h))$, but it remains to determine the action of the tracial state and the order on $K_0(C^*(\mathbf{Z}, M, h))$.

Identify $\overline{\eta}_1, \dots, \overline{\eta}_4$ with their images in $K_0(C^*(\mathbf{Z}, M, h))$. Let τ be the unique tracial state on $C^*(\mathbf{Z}, M, h)$, which comes from normalized Lebesgue measure μ on $(S^1)^3$. (See the beginning of the proof.) Clearly $\tau_*(\overline{\eta}_1) = 1$. Naturality in the Künneth formula for $S^1 \times S^1$ shows that the image under any point evaluation of $[z] \otimes [z] \in K_0(C(S^1 \times S^1))$ is zero. Therefore $[z] \otimes [z]$ can be represented as a difference $[p] - [q]$ of the classes of two projections of the same rank. (Actually, $[z] \otimes [z]$ is a Bott element, but we do not need this much.) The same is therefore true of η_2, η_3 and η_4, whence $\tau_*(\overline{\eta}_2) = \tau_*(\overline{\eta}_3) = \tau_*(\overline{\eta}_4) = 0$.

The calculations above show that $K^1(M)^h = \mathbf{Z}\gamma_1 + \mathbf{Z}\gamma_4$, so our next step is to calculate $\rho_h^\mu(\gamma_1)$ and $\rho_h^\mu(\gamma_4)$ as in Definition 1.2. We have $\gamma_1 = [z \otimes 1 \otimes 1]$ and

$$(z \otimes 1 \otimes 1) \circ h^{-1} = \exp(-2\pi i\theta)(z \otimes 1 \otimes 1),$$

so $\rho_h^\mu(\gamma_1) = \exp(2\pi i\theta)$ is immediate. For γ_4, write $[z] \otimes [z] = [p] - [q]$ as above. Working in a suitable matrix algebra, we see that γ_4 is represented by the unitary

$$[(1-p) \otimes 1 + p \otimes z][(1-q) \otimes 1 + q \otimes z^{-1}].$$

Its determinant is the constant function 1, so $\rho_h^\mu(\gamma_4) = 1$.

Choose $\nu_1^{(0)}, \nu_4^{(0)} \in K_0(C^*(\mathbf{Z}, M, h))$ such that $\partial(\nu_1^{(0)}) = \gamma_1$ and $\partial(\nu_4^{(0)}) = \gamma_4$. Theorem 1.3 implies that

$$\tau_*(\nu_1^{(0)}) \in \theta + \mathbf{Z} \quad \text{and} \quad \tau_*(\nu_4^{(0)}) \in \mathbf{Z}.$$

Taking $\nu_1 = \nu_1^{(0)} - k\overline{\eta}_1$ and $\nu_4 = \nu_4^{(0)} - l\overline{\eta}_1$ for suitable k and l, we get $\tau_*(\nu_1) = \theta$ and $\tau_*(\nu_4) = 0$.

We can now identify $K_0(C^*(\mathbf{Z}, M, h))$ as

$$\mathbf{Z}\overline{\eta}_1 \oplus \mathbf{Z}\overline{\eta}_4 \oplus \mathbf{Z}\nu_1 \oplus \mathbf{Z}\nu_4 \oplus (\mathbf{Z}/m\mathbf{Z}) \cdot \overline{\eta}_2 \oplus (\mathbf{Z}/n\mathbf{Z}) \cdot \overline{\eta}_3,$$

with

$$\tau_*(\overline{\eta}_1) = 1, \quad \tau_*(\nu_1) = \theta \quad \text{and} \quad \tau_*(\overline{\eta}_2) = \tau_*(\overline{\eta}_3) = \tau_*(\overline{\eta}_3) = \tau_*(\nu_4) = 0.$$

This is the required formula for τ_*, and the identification of $K_0(C^*(\mathbf{Z}, M, h))_+$ follows from Theorem 4.5(1) of [27]. ∎

In Section 6.1 of [12], it is stated without proof that if $|m| \neq |n|$ and $\theta \in \mathbf{R} \setminus \mathbf{Q}$, then the homeomorphisms $h_{m,n,\theta}$ and $h_{n,m,\theta}$, as defined in Lemma 3.1, are not flip conjugate. (The difference is that m and n have been exchanged.) We prove this, and in fact a more general statement, by computing the action on singular cohomology. The generalization was suggested by Qing Lin.

We start with the following lemma.

LEMMA 3.2: *Let*

$$a_1 = \begin{pmatrix} 1 & m_1 & r_1 \\ 0 & 1 & n_1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & m_2 & r_2 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix}$$

be integer matrices, with m_1, n_1, m_2 and n_2 all nonzero. Suppose a_1 is similar over $\mathbf{Z}$ to a_2 or to a_2^{-1}. Then $|m_1| = |m_2|$ and $|n_1| = |n_2|$.

Proof: First assume a_1 is similar over $\mathbf{Z}$ to a_2. Set $c_j = a_j - 1$. Then c_1 is similar over $\mathbf{Z}$ to c_2. A calculation shows that

$$c_j(\mathrm{Ker}(c_j^2)) = m_j \mathbf{Z} \cdot (1, 0, 0) = m_j \,\mathrm{Ker}(c_j).$$

Therefore $|m_1| = |m_2|$. Another calculation shows that

$$\mathbf{Z}^3 / c_j^2(\mathbf{Z}^3) \cong \mathbf{Z}/m_j n_j \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}.$$

So $|m_1 n_1| = |m_2 n_2|$. Since all four numbers are nonzero and $|m_1| = |m_2|$, it follows that $|n_1| = |n_2|$.

Now suppose that a_1 is similar over $\mathbf{Z}$ to a_2^{-1}. Since

$$a_2^{-1} = \begin{pmatrix} 1 & -m_2 & m_2 n_2 - r_2 \\ 0 & 1 & -n_2 \\ 0 & 0 & 1 \end{pmatrix},$$

the case already considered implies $|m_1| = |m_2|$ and $|n_1| = |n_2|$ in this case as well. ∎

LEMMA 3.3: *Let $M = (S^1)^3$. Let $m_1, n_1, m_2, n_2 \in \mathbf{Z} \setminus \{0\}$. For $j = 1, 2$ define $h_j = h_{m_j, n_j, \theta}\colon M \to M$ by*

$$h_j(\zeta_1, \zeta_2, \zeta_3) = (\exp(2\pi i \theta)\zeta_1, \zeta_1^{m_j}\zeta_2, \zeta_2^{n_j}\zeta_3)$$

for $(\zeta_1, \zeta_2, \zeta_3) \in (S^1)^3$. If h_1 is flip conjugate to h_2 then $|m_1| = |m_2|$ and $|n_1| = |n_2|$.

Proof: If $h_{m_1, n_1, \theta}$ is conjugate to $h_{m_2, n_2, \theta}$, then, in particular, there must be an automorphism b of $H^1((S^1)^3; \mathbf{Z})$ such that $b \circ (h_{m_1, n_1, \theta})^* \circ b^{-1} = (h_{m_2, n_2, \theta})^*$.

If instead $h_{m_1,n_1,\theta}$ is conjugate to $(h_{m_2,n_2,\theta})^{-1}$, then we get a similar equation with $[(h_{m_2,n_2,\theta})^*]^{-1}$ on the right. We show that no such b can exist.

We do the calculation of $(h_{m,n,\theta})^*$ in the proof of Lemma 3.1, but on singular cohomology rather than K-theory, and using the Künneth formula for singular cohomology, Theorem 5.6.1 of [36]. We get $H^1((S^1)^3;\mathbf{Z}) \cong \mathbf{Z}^3$, with a $\mathbf{Z}$-basis with respect to which the matrix of $(h_{m,n,\theta})^*$ is

$$(h_{m,n,\theta})^* = \begin{pmatrix} 1 & m & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

The nonexistence of b now follows from Lemma 3.2. ∎

Example 3.4: Fix $\theta \in [0,1] \setminus \mathbf{Q}$ and $m, n \in \mathbf{Z}$ with $0 < m < n$. Then the two affine Furstenberg transformations on $(S^1)^3$, given by

$$(\zeta_1, \zeta_2, \zeta_3) \mapsto (\exp(2\pi i\theta)\zeta_1, \zeta_1^m\zeta_2, \zeta_2^n\zeta_3)$$

and

$$(\zeta_1, \zeta_2, \zeta_3) \mapsto (\exp(2\pi i\theta)\zeta_1, \zeta_1^n\zeta_2, \zeta_2^m\zeta_3)$$

(the difference is that m and n have been exchanged), are not topologically orbit equivalent but have isomorphic crossed product C*-algebras.

They are not flip conjugate by Lemma 3.3, so not topologically orbit equivalent by Theorem 3.1 and Remark 3.4 of [3], or by Proposition 5.5 of [19]. The Elliott invariants are isomorphic by Lemma 3.1, and both are uniquely ergodic minimal diffeomorphisms whose tracial states induce maps from K_0 to $\mathbf{R}$ with dense ranges (namely $\mathbf{Z} + \theta\mathbf{Z}$). Therefore the two crossed products are isomorphic by Theorem 1.4.

From Proposition 1.12, we see that the algebra has two nonconjugate diagonals (Cartan subalgebras), both isomorphic to $C((S^1)^3)$.

We do not know whether these homeomorphisms are flip flow equivalent in the sense of Definition 1.6.

Example 3.5: Let $\theta \in [0,1] \setminus \mathbf{Q}$, let $r \in \mathbf{N}$ and let $p_1, p_2, \ldots, p_r$ be distinct primes. For $0 \leq k \leq r$ set

$$m_k = p_1 p_2 \cdots p_k \quad \text{and} \quad n_k = p_{k+1} p_{k+2} \cdots p_r.$$

By the same reasoning as in Example 3.4, the homeomorphisms (in the notation of Lemma 3.1) $h_{m_0,n_0,\theta}, h_{m_1,n_1,\theta}, \ldots, h_{m_r,n_r,\theta}$ are pairwise not topologically orbit equivalent, but, since

$$\mathbf{Z}/m_k\mathbf{Z} \oplus \mathbf{Z}/n_k\mathbf{Z} \cong \mathbf{Z}/p_1p_2\cdots p_r\mathbf{Z}$$

for all k, all give isomorphic crossed products. The common crossed product has $r+1$ nonconjugate diagonals (Cartan subalgebras), all isomorphic to $C((S^1)^3)$.

4. Third example: Minimal diffeomorphisms on distinct three dimensional manifolds

We produce minimal diffeomorphisms of $S^2 \times S^1$ and of $(S^1)^3$ whose crossed product C*-algebras are isomorphic. These are manifolds of the same dimension, but we will see that it is not possible for a minimal diffeomorphism on $S^2 \times S^1$ to be flip flow equivalent to a minimal diffeomorphism on $(S^1)^3$, let alone flip conjugate. In terms of three additonal ways to think of the results:

- The homotopy type of the space on which the homeomorphism acts is not an invariant of the transformation group C*-algebra, even if the dimension of the space is fixed.
- The C*-algebra in the example has two nonisomorphic diagonals (Cartan subalgebras).
- We do not know if the minimal diffeomorphism of $S^2 \times S^1$ can be chosen to be tempered in the sense of Definition 3.1 of [26], but it seems reasonable to hope that it can be.

LEMMA 4.1: *Let X be a connected compact metric space. Let $u \in M_n(C(X))$ be unitary. For $n \in \mathbf{N}$ let $h_n\colon X \to X$ be a homeomorphism such that $h_n^*([u]) = [u]$ in $K^1(X)$, and let $h\colon X \to X$ be a homeomorphism such that $h_n \to h$ uniformly. For each $n \in \mathbf{N}$ let μ_n be an h_n-invariant Borel probability measure on X, and set $\lambda_n = \rho_{h_n}^{\mu_n}([u])$. (See Definition 1.2.) Suppose $\lim_{n\to\infty} \lambda_n = \lambda$. Then there exists an h-invariant Borel probability measure μ on X such that $\rho_h^{\mu}([u]) = \lambda$.*

Proof: It is clear from Definition 1.2 that the rotation number is the same for the function $x \mapsto \det(u(x))$ from X to S^1. Thus, we may assume that u itself is a unitary in $C(X)$. Definition 1.2 then means that there are continuous functions $a_n\colon X \to \mathbf{R}$ such that

$$(u \circ h_n^{-1})^* u = \exp(2\pi i a_n) \quad \text{and} \quad \exp\left(2\pi i \int_X a_n d\mu_n\right) = \lambda_n.$$

Uniform convergence of h_n to h implies that $\lim_{n\to\infty} \|f \circ h_n - f \circ h\| = 0$ for all $f \in C(X)$. Furthermore, we claim that $\lim_{n\to\infty} \|f \circ h_n^{-1} - f \circ h^{-1}\| = 0$ for all $f \in C(X)$. Indeed, with $g = f \circ h^{-1}$, we get

$$\|f \circ h_n^{-1} - f \circ h^{-1}\| = \|g \circ h \circ h_n^{-1} - g\| = \|(g \circ h - g \circ h_n) \circ h_n^{-1}\| \to 0.$$

The set of Borel probability measures on X is a weak* compact subset of the dual of $C(X)$, by Alaoglu's Theorem, and it is weak* metrizable because it is bounded and $C(X)$ is separable. By passing to a subsequence, we may therefore assume that there is a Borel probability measure μ on X such that $\mu_n \to \mu$ weak*.

We claim that μ is h-invariant, and we prove this by showing that

$$\int_X (f \circ h) d\mu = \int_X f d\mu$$

for all $f \in C(X)$. So let $f \in C(X)$. We estimate:

$$\begin{aligned} &\left| \int_X (f\circ h) d\mu - \int_X f d\mu \right| \\ &\quad \leq \left| \int_X (f \circ h) d\mu - \int_X (f \circ h) d\mu_n \right| + \left| \int_X (f \circ h) d\mu_n - \int_X (f \circ h_n) d\mu_n \right| \\ &\quad\quad + \left| \int_X (f \circ h_n) d\mu_n - \int_X f d\mu_n \right| + \left| \int_X f d\mu_n - \int_X f d\mu \right|. \end{aligned}$$

The third term is zero because μ_n is h_n-invariant, the first and last terms converge to zero because $\mu_n \to \mu$ weak*, and the second term converges to zero because $\lim_{n\to\infty} \|f \circ h_n - f \circ h\| = 0$. This proves the claim.

Now set

$$w_n(x) = u(h_n^{-1}(x))^* u(x) \quad \text{and} \quad w(x) = u(h^{-1}(x))^* u(x) = \lim_{n\to\infty} w_n(x)$$

(uniform convergence). We know that w_n is homotopically trivial in $U(C(X))$. So w is also. Therefore there is a continuous function b: $X \to \mathbf{R}$ such that $w = \exp(2\pi i b)$.

Fix $x_0 \in X$. Then $\exp(2\pi i a_n(x_0)) \to \exp(2\pi i b(x_0))$. Therefore there are integers k_n such that $a_n(x_0) + k_n \to b(x_0)$. Define $b_n = a_n + k_n \in C(X)$. We claim that $\|b_n - b\| \to 0$.

To prove the claim, first notice that if $\zeta_1, \zeta_2 \in S^1$ with $|\zeta_1 - \zeta_2| < 1$, then the length of the arc from ζ_1 to ζ_2 is less than

$$|\mathrm{Re}(\zeta_1) - \mathrm{Re}(\zeta_2)| + |\mathrm{Im}(\zeta_1) - \mathrm{Im}(\zeta_2)| \leq 2|\zeta_1 - \zeta_2|.$$

Therefore if $\exp(2\pi i \alpha_j) = \zeta_j$ for $j = 1, 2$, then there is a unique integer k such that $|\alpha_1 + k - \alpha_2| < 1/\pi$; in fact, one gets $|\alpha_1 + k - \alpha_2| \leq (1/\pi)|\zeta_1 - \zeta_2|$. Now, dropping the initial terms of the sequence, we may assume that

$$\|w_n - w\| < 1 \quad \text{and} \quad |b_n(x_0) - b(x_0)| < 1/\pi$$

for all n. For any $x \in X$ and $n \in \mathbf{N}$, let $l_n(x)$ be the unique integer such that $|b_n(x) + l_n(x) - b(x)| < 1/\pi$. Since

$$|b_n(x) + l_n(x) - b(x)| \leq (1/\pi)|w_n(x) - w(x)| \leq (1/\pi)\|w_n - w(x)\|$$

by the above, we see that $b_n + l_n \to b$ uniformly. Further, for each n and as l runs through $\mathbf{Z}$, the sets

$$\{x \in X\colon l_n(x) = l\} = \{x \in X\colon |b_n(x) + l - b(x)| < 1/\pi\}$$

are disjoint, open and cover X. Since X is connected, only one can be nonempty, necessarily the one for $l = 0$. So $\|b_n - b\| \to 0$, and the claim is proved.

We now claim that

$$\lim_{n\to\infty} \int_X b_n d\mu_n = \int_X b d\mu.$$

Using the weak* convergence $\mu_n \to \mu$ on the second term at the second step, we have

$$\left| \int_X b_n d\mu_n - \int_X b d\mu \right| \leq \left| \int_X (b_n - b) d\mu_n \right| + \left| \int_X b d\mu_n - \int_X b d\mu \right| \to 0$$

as $n \to \infty$. This proves the claim.

Now

$$\lambda = \lim_{n\to\infty} \exp\left(2\pi i \int_X b_n d\mu_n\right) = \exp\left(2\pi i \int_X b d\mu\right),$$

which, by definition, is $\rho_h^\mu([u])$. ∎

Lemma 4.2: *Let M be a connected compact manifold such that $H^1(M; \mathbf{Z}) = 0$. Then the group $[M \times S^1, S^1]$ of homotopy classes (with operation given by pointwise multiplication on the codomain) is isomorphic to $\mathbf{Z}$, and is generated by the class of the function $u(x, \zeta) = \zeta$.*

Proof: First, note that $M \times S^1$, being a compact smooth manifold, is a finite CW complex. (See Theorem 3.5 and Corollary 6.7 of [22].) Therefore Theorems 8.1.8 and 8.1.1 of [36] (with π^Y being defined in Section 1.3 of [36]) show that $[M \times S^1, S^1] \cong H^1(M \times S^1; \mathbf{Z})$. We compute $H^1(M \times S^1; \mathbf{Z})$ using the Künneth formula for singular cohomology, Theorem 5.6.1 of [36]. This is allowed because $\mathbf{Z}$ is finitely generated and $H^*(S^1;\mathbf{Z})$ has finite type. Since $H^*(S^1;\mathbf{Z})$ is free and $H^1(M;\mathbf{Z}) = 0$, the result is just

$$H^1(M \times S^1;\mathbf{Z}) \cong H^0(M;\mathbf{Z}) \otimes H^1(S^1;\mathbf{Z}) \cong H^0(M;\mathbf{Z}) \cong \mathbf{Z}.$$

One checks, using the formulas for the maps in [36], that u corresponds to a generator of $H^1(M \times S^1;\mathbf{Z})$. ∎

PROPOSITION 4.3: *Let M be a connected compact manifold such that $H^1(M;\mathbf{Z}) = 0$. Define a unitary $u \in C(M \times S^1)$ by $u(x,\zeta) = \zeta$. Then there exists a uniquely ergodic minimal diffeomorphism h of $M \times S^1$ which is homotopic to the identity map and such that, with μ being the unique invariant Borel probability measure, $\rho_h^\mu([u]) \in S^1$ is equal to* $\exp(2\pi i\theta)$ *for some $\theta \in \mathbf{R} \setminus \mathbf{Q}$.*

Proof: Let $\mathrm{Diff}(M \times S^1)$ be the set of all C^∞ diffeomorphisms of $M \times S^1$, and, following Section 5 of [8], give it the C^∞ topology. For $\lambda \in S^1$, define $r_\lambda \colon M \times S^1 \to M \times S^1$ by $r_\lambda(x,\zeta) = (x, \lambda\zeta)$. This defines a free smooth action of S^1 on $M \times S^1$. Now let $D \subset \mathrm{Diff}(M \times S^1)$ be the closure of the set

$$D_0 = \{g \circ r_\lambda \circ g^{-1} : g \in \mathrm{Diff}(M \times S^1), \lambda \in S^1\}.$$

Fathi and Herman show that the subset S of D consisting of those elements which are minimal is a dense G_δ-set in D (5.6 of [8]), and that the subset T of D consisting of those elements which are uniquely ergodic is a dense G_δ-set in D (6.5 of [8]). All elements of D are homotopic to the identity map, because the r_λ are. Since D can be given a complete metric, it therefore suffices to show that the set of $h \in D$, for which there is some invariant Borel probability measure μ with $\rho_h^\mu([u]) \in S^1 \setminus \exp(2\pi i\mathbf{Q})$, is a dense G_δ-set in D.

For $\lambda \in S^1$, let $E_\lambda \subset D$ consist of those $h \in D$ for which there is an invariant Borel probability measure μ with $\rho_h^\mu([u]) = \lambda$. Lemma 4.1 implies that E_λ is closed. We show that its complement is dense.

Let $\omega \in S^1$. Write $\omega = \exp(2\pi i\alpha)$ with $\alpha \in \mathbf{R}$. Let a be the constant function $a(x,\zeta) = \alpha$ for all $(x,\zeta) \in M \times S^1$. Then

$$u(r_\omega^{-1}(x,\zeta))^* u(x,\zeta) = \exp(2\pi i a(x,\zeta)).$$

For any invariant measure μ, we can then compute $\rho_{r_\omega}^\mu([u])$ as

$$\rho_{r_\omega}^\mu([u]) = \exp\left(2\pi i \int_X a\,d\mu\right) = \exp(2\pi i\alpha) = \omega.$$

Now consider $\rho_h^\nu([u])$ with $h = g \circ r_\omega \circ g^{-1}$ for some $g \in \mathrm{Diff}(M \times S^1)$. We can write

$$[(u \circ g^{-1})(g \circ r_\omega^{-1} \circ g^{-1}(x,\zeta))]^*(u \circ g^{-1})(x,\zeta) = \exp(2\pi i(a \circ g^{-1})(x,\zeta)) = \omega,$$

so that $\rho_h^\nu([u \circ g^{-1}]) = \omega$ for any Borel probability measure ν which is invariant under h. It follows from Lemma 4.2 that $[u \circ g^{-1}] = [u]$ or $[u \circ g^{-1}] = [u^{-1}]$. Therefore $\rho_h^\nu([u]) \in \{\omega, \omega^{-1}\}$. In particular, if $\omega \notin \{\lambda, \lambda^{-1}\}$, then

$$g \circ r_\omega \circ g^{-1} \notin E_\lambda.$$

Since $\{r_\omega\colon \omega \in S^1 \setminus \{\lambda, \lambda^{-1}\}\}$ is clearly dense in $\{r_\omega\colon \omega \in S^1\}$, it follows that $D \setminus E_\lambda$ is dense in D, as claimed.

Since D has a complete metric, the set $\bigcap_{\lambda \in \mathbf{Q}}(D \setminus E_\lambda)$ is a dense G_δ-set in D. Moreover, its intersection

$$S \cap T \cap \bigcap_{\lambda \in \mathbf{Q}} (D \setminus E_\lambda)$$

with the dense G_δ-sets S and T from the beginning of the proof is again a dense G_δ-set, and in particular not empty. But any element of this set is a minimal diffeomorphism of $M \times S^1$ which satisfies the conclusion of the proposition. ∎

It seems reasonable to expect that, for any compact manifold M and any $\theta \notin \mathbf{Q}$, there is a uniquely ergodic minimal diffeomorphism such that the rotation number of $[u]$ with respect to its unique invariant measure is θ. Proving this, however, requires more work.

Example 4.4: Let $M_1 = S^2 \times S^1$, and let $u \in U(C(M_1))$ be given by $u(x,\zeta) = \zeta$. Use Proposition 4.3 to choose $\theta \in [0,1] \setminus \mathbf{Q}$ and a uniquely ergodic minimal diffeomorphism $h_1\colon M_1 \to M_1$, with unique invariant Borel probability measure μ, which is homotopic to the identity map and such that $\rho_{h_1}^\mu([u]) = \exp(2\pi i\theta) \in S^1$. Let $M_2 = (S^1)^3$, and define $h_2\colon M_2 \to M_2$ by

$$h_2(\zeta_1, \zeta_2, \zeta_3) = (\exp(2\pi i\theta)\zeta_1, \zeta_1\zeta_2, \zeta_2\zeta_3)$$

for $(\zeta_1, \zeta_2, \zeta_3) \in (S^1)^3$. We show that $C^*(\mathbf{Z}, M_1, h_1) \cong C^*(\mathbf{Z}, M_2, h_2)$. Note that h_1 can't be topologically orbit equivalent to h_2, because M_1 is not homeomorphic to M_2.

Lemma 3.1 implies that h_2 is minimal and uniquely ergodic, and computes the Elliott invariant of $C^*(\mathbf{Z}, M_2, h_2)$. We calculate the Elliott invariant of $C^*(\mathbf{Z}, M_1, h_1)$.

The Künneth Theorem (Theorem 4.1 of [34]; see also Corollary 2.7.15 of [1] for the commutative case, which suffices here) shows that

$$K^*(S^2 \times S^1) \cong K^*(S^2) \otimes K^*(S^1).$$

We identify the two sides of this isomorphism. Let $1 \in C(S^2)$ be the identity, and let $\beta \in K^0(S^2)$ be the Bott element, which is of the form $[p] - [q]$ for rank one projections $p, q \in M_2(C(S^2))$. Let $1 \in C(S^1)$ be the identity, and let $z \in C(S^1)$ be the canonical unitary $z(\zeta) = \zeta$. Then $K^0(S^2 \times S^1)$ is the free abelian group on generators

$$\eta_1 = [1] \otimes [1] \quad \text{and} \quad \eta_2 = \beta \otimes [1],$$

and $K^1(S^2 \times S^1)$ is the free abelian group on generators

$$\gamma_1 = [1] \otimes [z] = [u] \quad \text{and} \quad \gamma_2 = \beta \otimes [z].$$

Since h_1 is homotopic to the identity map (by construction), $h_1^* = \mathrm{id}$, and the exact sequence of Theorem 1.1 breaks apart into the two exact sequences

$$0 \longrightarrow \mathbf{Z}\eta_1 + \mathbf{Z}\eta_2 \longrightarrow K_0(C^*(\mathbf{Z}, M_1, h_1)) \xrightarrow{\partial} \mathbf{Z}\gamma_1 + \mathbf{Z}\gamma_2 \longrightarrow 0$$

and

$$0 \longrightarrow \mathbf{Z}^2 \longrightarrow K_1(C^*(\mathbf{Z}, M_1, h_1)) \longrightarrow \mathbf{Z}^2 \longrightarrow 0.$$

Both split because $\mathbf{Z}^2$ is free. Therefore

$$K_0(C^*(\mathbf{Z}, M_1, h_1)) \cong \mathbf{Z}^4 \cong K_0(C^*(\mathbf{Z}, M_2, h_2))$$

and

$$K_1(C^*(\mathbf{Z}, M_1, h_1)) \cong \mathbf{Z}^4 \cong K_1(C^*(\mathbf{Z}, M_2, h_2)).$$

We are done with $K_1(C^*(\mathbf{Z}, M_1, h_1))$, but it remains to determine the action of the tracial state and the order on $K_0(C^*(\mathbf{Z}, M_1, h_1))$.

Identify η_1 and η_2 with their images in $K_0(C^*(\mathbf{Z}, M_1, h_1))$. Let τ be the unique tracial state on $C^*(\mathbf{Z}, M_1, h_1)$, which comes from the unique ergodic measure μ on M_1. Clearly $\tau_*(\eta_1) = 1$. We have $\tau_*(\eta_2) = 0$ because $\beta = [p \otimes 1] - [q \otimes 1]$ and p and q have the same rank at each point.

We have $K^1(M_1)^{h_1} = K^1(M_1)$ by the above, and our next step is to calculate $\rho_h^\mu(\gamma_1)$ and $\rho_h^\mu(\gamma_2)$ as in Definition 1.2. We have $\rho_h^\mu(\gamma_1) = \exp(2\pi i\theta)$ by construction. For γ_2, write $\beta = [p] - [q]$ as above. Working in a suitable matrix algebra, we see that γ_2 is represented by the unitary

$$[(1-p) \otimes 1 + p \otimes z][(1-q) \otimes 1 + q \otimes z^{-1}].$$

Its determinant is the constant function 1, so $\rho_h^\mu(\gamma_2) = 1$.

Choose $\nu_1^{(0)}, \nu_2^{(0)} \in K_0(C^*(\mathbf{Z}, M_1, h_1))$ such that $\partial(\nu_1^{(0)}) = \gamma_1$ and $\partial(\nu_2^{(0)}) = \gamma_2$. Theorem 1.3 therefore implies that

$$\tau_*(\nu_1^{(0)}) \in \theta + \mathbf{Z} \quad \text{and} \quad \tau_*(\nu_2^{(0)}) \in \mathbf{Z}.$$

Taking $\nu_1 = \nu_1^{(0)} - k\overline{\eta}_1$ and $\nu_2 = \nu_2^{(0)} - l\overline{\eta}_1$ for suitable k and l, we get $\tau_*(\nu_1) = \theta$ and $\tau_*(\nu_2) = 0$.

We can now identify $K_0(C^*(\mathbf{Z}, M_1, h_1))$ as

$$\mathbf{Z}\eta_1 \oplus \mathbf{Z}\eta_2 \oplus \mathbf{Z}\nu_1 \oplus \mathbf{Z}\nu_2,$$

with

$$\tau_*(\eta_1) = 1, \quad \tau_*(\nu_1) = \theta \quad \text{and} \quad \tau_*(\eta_2) = \tau_*(\nu_4) = 0.$$

Comparing this with the result of Lemma 3.1 for the case $m = n = 1$, we see that there is an isomorphism $f\colon K_0(C^*(\mathbf{Z}, M_1, h_1)) \to K_0(C^*(\mathbf{Z}, M_2, h_2))$ which preserves the tracial states and the class of the identity. We already have the isomorphism on K_1, and the range of the tracial state on K_0 is dense, so Theorem 1.4 implies that $C^*(\mathbf{Z}, M_1, h_1) \cong C^*(\mathbf{Z}, M_2, h_2)$.

From Proposition 1.12, we see that the algebra has a diagonal (Cartan subalgebra) isomorphic to $C(M_1)$, and another one isomorphic to $C(M_2)$.

Lemma 4.5: *The minimal diffeomorphisms h_1 and h_2 of Example 4.4 are not flip flow equivalent.*

Proof: It follows from Theorem 2 of [35] that if h_1 and h_2 are homeomorphisms of connected compact metric spaces X_1 and X_2, and if h_1 and h_2 are flip flow equivalent, then the universal covers of X_1 and X_2 are homeomorphic. Clearly, however, the universal covers of $S^2 \times S^1$ and $(S^1)^3$ are not homeomorphic. ∎

5. Fourth example: Minimal diffeomorphisms on manifolds of different dimensions

We find uniquely ergodic minimal diffeomorphisms $h_n\colon S^n \times S^1 \to S^n \times S^1$, for $n \geq 3$ odd, such that the crossed product C*-algebras $C^*(\mathbf{Z}, S^n \times S^1, h_n)$ are all isomorphic. No two of these diffeomorphisms can be topologically conjugate, or even flip flow equivalent, since they act on manifolds of different dimensions. In terms of the three additional ways to think of the results:

- The dimension of the space on which the homeomorphism acts is not an invariant of the transformation group C*-algebra.

- The C*-algebra in the example has infinitely many diagonals (Cartan subalgebras) which are far from being isomorphic to each other, having maximal ideal spaces of different dimensions.
- We do not know if the minimal diffeomorphisms in this example can be chosen to be tempered in the sense of Definition 3.1 of [26], but it seems reasonable to hope that they can be. If so, this example seems to be a good candidate for one in which the smooth crossed products are not isomorphic.

The basis of the construction is the existence of uniquely ergodic minimal diffeomorphisms on odd spheres and the following result, obtained by combining several results from [25].

LEMMA 5.1: *Let X be a connected compact metric space, and let $h\colon X \to X$ be a uniquely ergodic minimal homeomorphism. Then there is a dense G_δ-set $T \subset S^1$ such that, for every $\lambda \in T$, the homeomorphism of $X \times S^1$ given by $(x, \zeta) \mapsto (h(x), \lambda\zeta)$ is minimal and uniquely ergodic.*

Proof: For $\lambda \in S^1$, define $r_\lambda\colon X \times S^1 \to X \times S^1$ by $r_\lambda(x, \zeta) = (x, \lambda\zeta)$. This is a continuous action of S^1 on $X \times S^1$. We claim that the pair $(X \times S^1, h \times \mathrm{id})$ is a simple free extension of (X, h) in the sense of [25] (see Equation (0.4)). Indeed, for $n \in \widehat{S^1} \cong \mathbf{Z}$, the required continuous function $f_n\colon X \times S^1 \to S^1$ can be taken to be simply $f_n(x, \zeta) = \zeta^n$.

Since $X \times S^1$ is connected, Theorem 2 of [25] implies that

$$\{\lambda \in S^1 : r_\lambda \circ (h \times \mathrm{id}) \text{ is minimal}\}$$

contains a dense G_δ-set in S^1. Since h is uniquely ergodic, we use the version of Theorem 4 of [25] indicated in Remark 3 (following Theorem 5) there, to show that

$$\{\lambda \in S^1 : r_\lambda \circ (h \times \mathrm{id}) \text{ is uniquely ergodic}\}$$

contains a dense G_δ-set in S^1. The set T of the lemma is obtained by intersecting these two sets. ∎

Example 5.2: Use Theorem 3 (in Section 3.8) of [8] to find, for each odd $n \geq 3$, a uniquely ergodic minimal diffeomorphism $h_n^{(0)}\colon S^n \to S^n$. We note that $h_n^{(0)}$ must have degree 1, since the Lefschetz fixed point theorem (Theorem 4.7.7 of [36]) implies that an orientation reversing diffeomorphism of an odd sphere must have a fixed point; see 4.7.9 of [36]. Therefore $h_n^{(0)}$ is homotopic to the identity map.

Let $T_n \subset S^1$ be the dense G_δ-set associated with $h_n^{(0)}$ from Lemma 5.1. Then $T = \bigcap_{k=1}^\infty T_{2k+1}$ is still a dense G_δ-set. Choose $\theta \in [0,1]$ such that $\exp(2\pi i\theta) \in T$. Define $h_n \colon S^n \times S^1 \to S^n \times S^1$ by $h_n(x,\zeta) = (h_n^{(0)}(x), \exp(2\pi i\theta)\zeta)$. Then each h_n is minimal and uniquely ergodic, by the choice of T. In particular, $\theta \notin \mathbf{Q}$. Also, each h_n is homotopic to the identity map. We prove that the C*-algebras $C^*(\mathbf{Z}, S^n \times S^1, h_n)$ are all isomorphic. No two of these diffeomorphisms can be topologically conjugate, or even flip flow equivalent, since they act on manifolds of different dimensions.

As usual, we start by using the Künneth Theorem (Theorem 4.1 of [34] or Corollary 2.7.15 of [1]), to get

$$K^*(S^n \times S^1) \cong K^*(S^n) \otimes K^*(S^1),$$

and we identify the two sides of this isomorphism. We write

$$K^0(S^n) = \mathbf{Z}\cdot[1], \quad K^1(S^n) = \mathbf{Z}\cdot\gamma, \quad K^0(S^1) = \mathbf{Z}\cdot[1] \quad \text{and} \quad K^1(S^1) = \mathbf{Z}\cdot[z],$$

with γ equal to the class of a suitable unitary in some matrix algebra over $C(S^n)$ and with $z \in C(S^1)$ given by $z(\zeta) = \zeta$. Then, suppressing the dependence on n in the notation, $K^0(S^n \times S^1)$ is the free abelian group on generators

$$\eta_1 = [1] \otimes [1] \quad \text{and} \quad \eta_2 = \gamma \otimes [z],$$

and $K^1(S^n \times S^1)$ is the free abelian group on generators

$$\gamma_1 = [1] \otimes [z] \quad \text{and} \quad \gamma_2 = \gamma \otimes [1].$$

The exact sequence of Theorem 1.1 breaks apart into the two exact sequences

$$0 \longrightarrow \mathbf{Z}\eta_1 + \mathbf{Z}\eta_2 \longrightarrow K_0(C^*(\mathbf{Z}, S^n \times S^1, h_n)) \xrightarrow{\partial} \mathbf{Z}\gamma_1 + \mathbf{Z}\gamma_2 \longrightarrow 0$$

and

$$0 \longrightarrow \mathbf{Z}^2 \longrightarrow K_1(C^*(\mathbf{Z}, S^n \times S^1, h_n)) \longrightarrow \mathbf{Z}^2 \longrightarrow 0.$$

Both split because $\mathbf{Z}^2$ is free, so (with obvious identifications, and with $\partial(\nu_j^{(0)}) = \gamma_j$)

$$K_0(C^*(\mathbf{Z}, S^n \times S^1, h_n)) = \mathbf{Z}\eta_1 + \mathbf{Z}\eta_2 + \mathbf{Z}\nu_1^{(0)} + \mathbf{Z}\nu_1^{(0)}$$

and

$$K_1(C^*(\mathbf{Z}, S^n \times S^1, h_n)) \cong \mathbf{Z}^4.$$

Let τ be the unique tracial state on $C^*(\mathbf{Z}, S^n \times S^1, h_n)$, which comes from the product $\mu \times \lambda$ of the unique ergodic measure μ for $h_n^{(0)}$ on S^n and normalized Haar measure λ on S^1. (This product is clearly h_n-invariant, and by construction there is only one invariant probability measure.) Clearly $\tau_*(\eta_1) = 1$. Since η_2 vanishes under point evaluations (by naturality in the Künneth Theorem), it follows that $\tau_*(\eta_2) = 0$. We have $K^1(M_1)^{h_1} = K^1(M_1)$ by the above, and our next step is to calculate $\rho_{h_n}^{\mu\times\lambda}(\gamma_1)$ and $\rho_{h_n}^{\mu\times\lambda}(\gamma_2)$ as in Definition 1.2. It is easy to check (using $n \geq 3$ and Corollary VI.12(i) of [7] for the second step) that

$$\rho_{h_n}^{\mu\times\lambda}(\gamma \otimes [1]) = \rho_{h_n^{(0)}}^{\mu}(\gamma) = 1,$$

and (with r being rotation by $\exp(2\pi i\theta)$) that

$$\rho_{h_n}^{\mu\times\lambda}([1] \otimes [z]) = \rho_r^{\lambda}([z]) = \exp(2\pi i\theta).$$

As usual, we define ν_1 and ν_2 by subtracting suitable multiples of [1] from $\nu_1^{(0)}$ and $\nu_2^{(0)}$. This gives

$$K_0(C^*(\mathbf{Z}, S^n \times S^1, h_n)) = \mathbf{Z}\eta_1 + \mathbf{Z}\eta_2 + \mathbf{Z}\nu_1 + \mathbf{Z}\nu_1$$

with

$$[1] = \eta_1, \quad \tau_*(\eta_1) = 1, \quad \tau_*(\nu_1) = \theta \quad \text{and} \quad \tau_*(\eta_2) = \tau_*(\nu_4) = 0.$$

The generators $\eta_1, \eta_2, \nu_1, \nu_1$ depend on n, but it is clear from this formula that for any two values of n there is an isomorphism between these groups which preserves τ_* and [1]. Since the range of the tracial state on K_0 is dense, Theorem 1.4 implies that the C*-algebras are pairwise isomorphic.

From Proposition 1.12, we see that an algebra in the common isomorphism class has, for every odd $n \geq 3$, a diagonal (Cartan subalgebra) isomorphic to $C(S^n \times S^1)$.

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